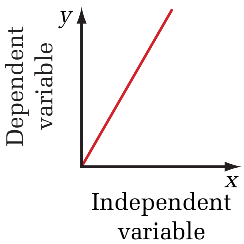
Mathematical Modelling and Curve Straightening

Patterns in quantitative data can be expressed in the form of mathematical equations. These relationships form a type of *mathematical model* of the phenomenon being studied. You can use the model to examine trends and to make testable numerical predictions.

Identifying Types of Relationships

Graphing data is a common way of revealing patterns. Simply by drawing a best-fit curve through the data points, it might be possible to identify a general type of mathematical relationship expressed in the observations. Four common patterns are illustrated below. Each pattern can be expressed algebraically as a proportionality statement ($a \propto b$) or as an equation. In mathematics courses, you might also have studied the graphs and equations of logarithmic, sinusoidal, or other types of relationships.

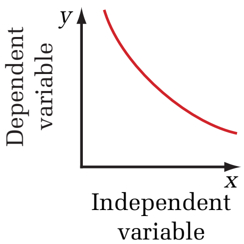
Basic mathematical relationships



A linear

$y \propto x$

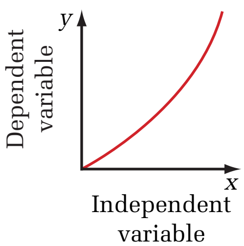
$y = kx$



B inverse

$y \propto \frac{1}{x^n}$ or $y \propto x^{-n}$

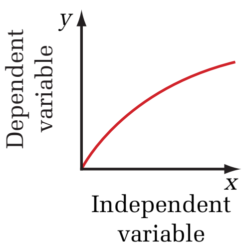
$y = kx^{-n}$



C exponential

$y \propto x^n$

$y = kx^n$



D root

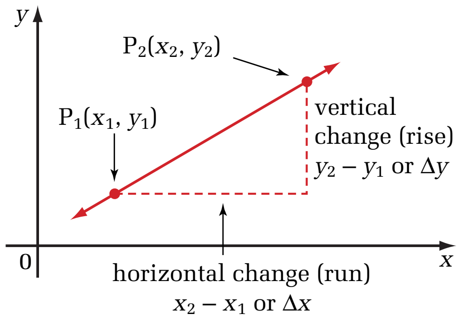
$y \propto \sqrt[n]{x}$ or $y \propto x^{\frac{1}{n}}$

$y = kx^{\frac{1}{n}}$

Linear Relationships

In previous studies, you have used the straight-line graph of a linear relationship to produce a specific mathematical equation that represents the graph. The equation is completely determined by the slope, m, of the graph and its y-intercept, b.

The equation of a straight-line graph



$$\text{slope } (m) = \frac{\text{vertical change (rise)}}{\text{horizontal change (run)}} = \frac{\Delta y}{\Delta x}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1},\ x_2 \neq x_1$$

Equation of the line: $y = mx + b$

Straightening Non-Linear Graphs

You can often produce a straight-line graph from a non-linear relationship by making an appropriate choice of independent variables for the graph. By analyzing the resulting straight line, you can obtain an equation that fits the data. This procedure, which is called "curve straightening," produces equations of the form

(quantity on the vertical axis =
m (quantity on the horizontal axis) + b

You can straighten a curve by selecting the quantity graphed on the horizontal axis to match the general type of variation shown by the data. If the independent variable is x and you suspect

- inverse variation: plot $\frac{1}{x}$ or $\frac{1}{x^2}$ or $\frac{1}{x^3}$ on the horizontal axis
- exponential variation: plot x^2 or x^3 on the horizontal axis
- root variation: plot $x^{\frac{1}{2}}$ or $x^{\frac{1}{3}}$ on the horizontal axis

There is no mathematical reason why other exponents could not be used. Most phenomena examined in this course, however, are best modelled using integer exponents or roots no greater than three.

Procedure

1. From a table of raw data for two variables, x and y, produce an initial graph of y versus x.
2. Identify the general type of relationship shown by the graph.
3. Modify the independent variable to suit the proposed type of relationship. Add the new quantity to your data table and then draw a new graph of y against this quantity derived from x.
4. If the new graph is a straight line, calculate its slope and y-intercept. Use these values to write and simplify an equation to represent the data.
5. If the new graph is not a straight line, repeat steps 3 and 4, using a different modification of the independent variable until you obtain a straight-line graph.

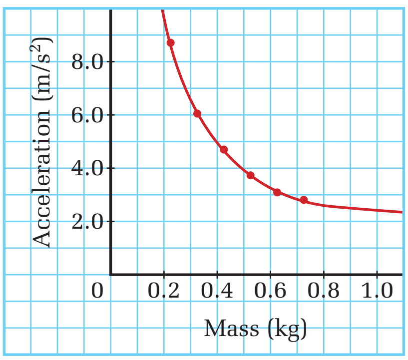
Example

A force of 1.96 N was used to accelerate a lab cart with mass 0.225 kg. The mass of the cart was then systematically increased, producing the accelerations shown below. Find an equation that represents this data.

Mass (kg)	Acceleration $\left(\frac{m}{s^2}\right)$
0.225	8.71
0.325	6.05
0.425	4.70
0.525	3.73
0.625	3.09
0.725	2.81

1. Graph the raw data.

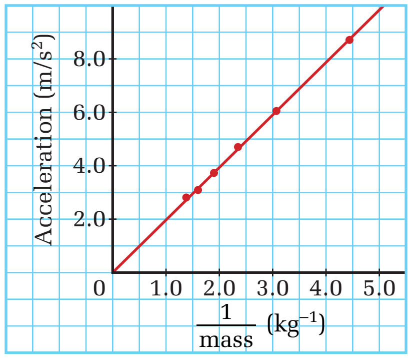
Acceleration against mass



2. Identify the type of variation. This graph shows inverse variation.
3. Modify the independent variable, extend the data table, and regraph the data. Choose the most simple possibility, $\frac{1}{\text{mass}}$, to investigate.

Mass (kg)	Acceleration $\left(\frac{m}{s^2}\right)$	$\frac{1}{\text{mass}}$ $\left(\frac{1}{\text{kg}}\right)$
0.225	8.71	4.44
0.325	6.05	3.07
0.425	4.70	2.35
0.525	3.73	1.90
0.625	3.09	1.60
0.725	2.81	1.38

Acceleration against $\frac{1}{\text{mass}}$



4. Since the graph is a straight line, use its slope and y-intercept to obtain its equation.

   slope (m) = 1.96

   y-intercept (b) 0.0500

   Equation of the line

   (quantity on the vertical axis) =
   m (quantity on the horizontal axis) + b

   $$\text{acceleration} = 1.96\left(\frac{1}{\text{mass}}\right) + 0.0500$$

   $$\text{acceleration} = \frac{1.96}{m} + 0.0500$$

5. If the graph had not been a straight line, the next most simple variation of the independent variable would have been considered: $\frac{1}{(\text{mass})^2}$.
6. The equation determined from the data is reasonable, because the situation is an example of Newton's second law, $F = ma$. Solved for acceleration, this becomes $a = \frac{F}{\text{m}}$,

Skill Set 4 continued from previous page

or $a = F\left(\frac{1}{m}\right)$, which has the same form as the equation for the graph. The slope of the graph represents the force applied to the cart.
The y-intercept, 0.0500, is probably due to experimental error, as the graph should pass through the origin.

Points to Remember

Make sure that the scales on your graph axes start at zero. Otherwise, you will see a magnified view of only a small portion of the graph. The overall shape of the graph might not be shown, so it will be difficult to identify the type of variation in the data. Part of a gentle curve, for example, can appear to be a straight line.

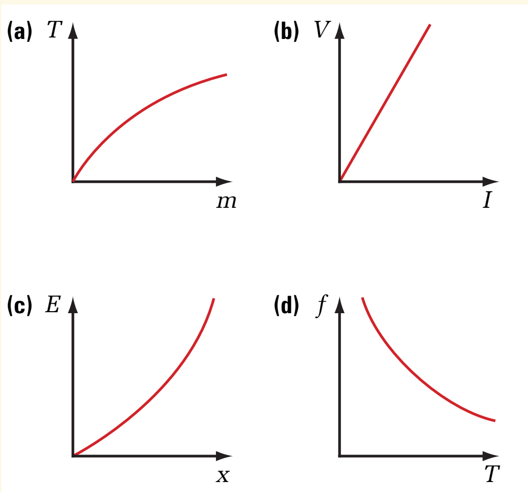
Sometimes it is a good idea to look at only part of a data set. The relationship between force applied to a spring and its extension (Hooke's law), for example, is linear, as long as the force does not exceed a certain value. Rather than trying to find a relationship that fits the entire graph, only the linear portion is usually considered. An initial graph of your data will show parts that are easy to model and will also reveal data points that are far from the best-fit line. Obvious errors are often excluded when developing equations to fit data.

Some computer programs are capable of automatic curve fitting. The resulting equations might fit experimental data well, without being very helpful. If you suspect that a certain phenomenon follows an inverse square law, for example, it would be sensible to choose $\frac{1}{(\text{dependent variable})^2}$ as the quantity to graph. Then you can determine how closely your observations approach an ideal model.

You might want to investigate other aspects of mathematical modelling. If you are familiar with logarithms, for example, consider the advantages and disadvantages of curve straightening by graphing, or the use of log or semi-log graph paper. You might also look into correlation coefficients, statistical measures that can express how well a given equation models a set of data. Finally, you might explore the use of power series to produce approximations of complex relationships.

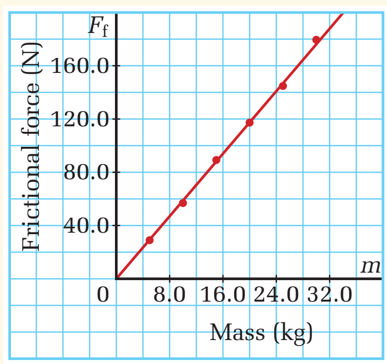
SET 4 Skill Review

1. Name the type of relationship represented by each graph below and write the relationship as a proportion and as a general equation.



2. The graph in the next column shows the force of static friction when you push different masses placed on a horizontal carpet and they start to move.
   (a) Determine the slope and intercept of the graph below.
   (b) Write an equation that represents the data.
   (c) Explain the physical meaning of each numerical coefficient in the equation.

Static friction between leather-soled shoes and a carpet



3. For each of the following relationships, use curve-straightening techniques to determine an equation that represents the data. If possible, validate your solution by giving physical reasons why the relationship must have the form it does.

(a) The gravitational attraction between two lead spheres in a Cavendish apparatus depends on the separation between their centres.

Separation (cm)	Gravitational force ($N \times 10^{-9}$)
55	3.13
70	1.93
85	1.31
100	0.946
115	0.715
130	0.560

(b) The buoyant force on a spherical weather balloon depends on how much the balloon is inflated (the volume of the balloon).

Radius of balloon (m)	Force (N)
2.285	569
2.616	855
2.879	1135
3.102	1422
3.296	1709
3.470	1993
3.628	2281

(c) The power dissipated by a light bulb is related to the electric current flowing through the light bulb.

Electrical current (A)	Power (W)
16.0	15
20.6	25
26.3	40
32.0	60
35.7	75
41.2	100

(d) The kinetic energy of a moving car depends on the car's velocity.

Velocity $\left(\frac{km}{h}\right)$	Kinetic energy (kJ)
15	15.4
25	42.7
35	83.8
45	139.4
55	208.9
65	289.8